Thermodynamics and Ice-Melt Flows
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Entropy and Compressing Liquid Water

The objective of these notes is to estimate the temperature increase in subcooled liquid water for an increase of about 1MPa in pressure, which corresponds to about 100 m change in column depth.

The estimate is first made by use of a computer program that calculates the thermodynamic, thermophysical, and transport properties of water. The computer program is the NBS/NRC formulation for water properties by Haar, Gallagher, and Kell [1984]. There are a multitude of that compute water properties, this is one version that I happen to have available.

If the dissipation of potential energy is assumed to be, as with the case of the papers discussed in the previous post,

$$\varepsilon_{PE} = gh \tag{1}$$

and the dissipation taken to be a change in enthalpy, the temperature change is approximately

$$\Delta T_{PE} = \frac{gh}{C_p} \tag{2}$$

or approximately 0.23 K for 100.0 m, as reported in Isenko, *et al*. [2005].

Calculation of the temperature change due to the work by gravity on the fluid is as follows. The calculation is summarized in the nearby Table 1. The initial state is taken to be pressure of 1.0 MPa and temperature of

288.000 K, shown at the top of the left-hand column of the table. The pressure at the final state is given by

$$P_2 = P_1 + \rho g h \tag{3}$$

For density of 1000.0 kg/m^3 and depth of 100.0 m, the pressure at the final state is 1.98025066 MPa. This value could be tuned up a bit, but that is not necessary.

The pressure and temperature at the initial state determines the entropy at the initial state, shown to be $s_1 = 0.221906$ kJ/kg K. The pressure at the final state is shown at the top of the right-hand column in the Table.

For isentropic compression by gravity, the final state is determined by the increased pressure and the entropy at the initial state, and these are shown in green in Table 1.

Property	Initial State	Final State
Pressure (MPa)	1.0	1.98066352
Temperature (K)	288.000	288.010
Internal Energy (kJ/kg)	62.2442	62.2450
Enthalpy (kJ/kg)	63.2447	64.2256
Entropy (kJ/kg K)	0.221906	0.221906
Density (kg/m^3)	999.579	1000.03
Elevation (m)	0.000	100.000
Cp (kJ/kg K)	4.18072	4.17732
Cv (kJ/kg K)	4.16638	4.16257
β_P (1/T)	0.152190	0.154389
κ_T (1/MPa)	4.65885x10^{-04}	4.64598x10^{-04}

Table 1. Compression of liquid water by gravity.

As shown in Table 1, isentropic compression by about 1MPa corresponds to a temperature change of about 0.010 C; the value in red in the right-hand column. The value given in the Isenko *et al*. paper, 0.2 C, corresponds to the value associated with the assumed viscous dissipation as by Eq. (2).

Another Estimate

The temperature change can also be estimated by use of the functional dependency of entropy on pressure and temperature

$$s = s(P,T) \tag{4}$$

with

$$\begin{aligned} ds = 0 &= \left(\frac{\partial s}{\partial P}\right)_T dP + \left(\frac{\partial s}{\partial T}\right)_P dT \\ \left(\frac{\partial s}{\partial T}\right)_P dT &= -\left(\frac{\partial s}{\partial P}\right)_T dP \end{aligned} \tag{5}$$

The Bridgman tables [1914, 1961] give

$$\begin{aligned} \left(\frac{\partial s}{\partial T}\right)_P &= \frac{(\partial s)_P}{(\partial T)_P} = \frac{C_P}{T} \\ \left(\frac{\partial s}{\partial P}\right)_T &= \frac{(\partial s)_T}{(\partial P)_T} = -\frac{\beta}{\rho} \end{aligned} \tag{6}$$

where β is the coefficient of thermal expansion

$$\beta = -\frac{1}{\rho}\left(\frac{\partial \rho}{\partial T}\right)_P \tag{7}$$

The water-properties program gives $\left(\frac{\partial P}{\partial \rho}\right)_T$ and $\left(\frac{\partial P}{\partial \rho}\right)_T$ as output, and Maxwell's equations; J. C Maxwell, ca. 1870; see also Callen [1960]. give

$$\left(\frac{\partial \rho}{\partial T}\right)_P = -\frac{\left(\frac{\partial P}{\partial T}\right)_\rho}{\left(\frac{\partial P}{\partial \rho}\right)_T} \tag{8}$$

Putting all this together gives the coefficient of thermal expansion at the initial state to be $\beta = 1.5225 \text{x} 10^{-4}$ 1/K

Taking the state and thermophysical properties to be constants, integrating the second of Eq. (5) gives

$$\ln\frac{T_2}{T_1} = -\frac{\beta}{\rho C_P}(P_2 - P_1) \tag{9}$$

and the temperature at the final state to be

$$T_2 = 288.0103 \text{ K} \tag{10}$$

which is in agreement with the results in Table 1 above.

References

Evgeni Isenko, Renji Naruse, and Bulat Mavlyudov, "Water temperature in englacial and supraglacial channels: Change along the flow and contribution to ice melting on the channel wall," *Cold Regions Science and Technology*, Vol. 42, pp. 53– 62, 2005.

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Herbert B. Callen, *Thermodynamics: An Introduction to the Physical Theories of Equilibrium Thermostatistics and Irreversible Thermodynamics*, John Wiley & Sons, Incorporated, New York, (1960).

P. W. Bridgman, "A Complete Collection of Thermodynamic Formulas," *Physical Review*, Vol. 3, No. 4, pp. 273–281, (1914). doi:10.1103/PhysRev.3.273.

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